Classical field theory 2012 (NS-364B) – Midterm

Tue Apr 17 2012, 15:00-18:00, MG Kantine; assistant: Laurent Dufour.
You have *three* hours to solve this midterm. The exam has in total 30 points (plus bonus points), and it carries 30% of the grade. The exam is closed books.
There are two *bonus questions*: 3E and 5D. If you solve them, you will get extra points.

Problem 1. Theoretical questions. *(6 points)*

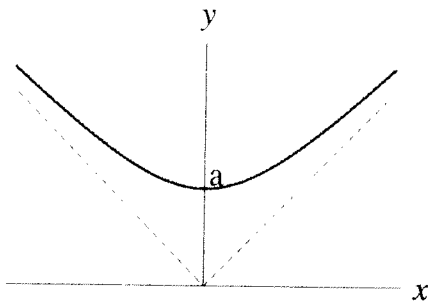
(A) State the Noether theorem of classical mechanics.

(B) Define the canonical Poisson bracket in field theory. Write down the Poisson algebra for the field $\phi_I(\vec{x},t)$ and its canonical momentum $\pi^J(\vec{y},t)$. *i.e.* the canonical Poisson brackets for the three possible pairs.

(C) There are two types of boundary conditions on a closed surface in electrostatics one can choose from: the Dirichlet and von Neumann boundary conditions (or a linear combination of the two). What is the Dirichlet and what von Neumann boundary condition? Can you provide an argument to why it is enough to specify one of the two conditions in order to uniquely solve for the potential in the region surrounded by a closed surface?

(D) The gyromagnetic ratio of a particle (or of a collection of particles) relates macroscopic quantities with microscopic quantities. What are these quantities?

Problem 2. Dynamics of constrained systems. *(6 points)*



Consider a particle moving on a hyperbola $y^2 - x^2 = a^2$ (see the figure above), where a is some distance scale, and whose Lagrangian is given by,

$$L = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2]\,. \tag{1}$$

(A) By making an appropriate substitution of coordinates (what is it?), show that the Lagragian (1) can be written as.

$$L = \frac{1}{2}ma^2\cosh[2\phi(t)]\dot{\phi}^2\,. \tag{2}$$

Calculate the canonical momentum p_ϕ and show that the Hamiltonian can be written as,

$$H = \frac{p_\phi^2}{2ma^2\cosh(2\phi)} \qquad p_\phi = ma^2\cosh(2\phi)\dot{\phi}\,. \tag{3}$$

(B) Solve for $\phi = \phi(t)$. You can write your solutions implicitly, *i.e.* in this case as $t = t(\phi)$. Show that for small/large ϕ, the solution for $\phi(t)$ can be written as (for $t > 0$),

$$\phi(t) \approx \frac{vt}{a} - \frac{(vt)^3}{3a^3}, \qquad \phi(t) \approx \ln\left(\frac{vt}{\sqrt{2}a} + \phi_0 + \sqrt{\left(\frac{vt}{\sqrt{2}a} + \phi_0\right)^2 + 1}\right), \tag{4}$$

where v is the (conserved) particle's speed and $\phi_0 = \Gamma^2(3/4)/\sqrt{4\pi} \simeq 0.599$ is a constant. Show also that for large times $t \gg a/v$,

$$x \approx y \approx \frac{vt}{\sqrt{2}} + a\phi_0 \tag{5}$$

Could you have guessed this answer (without doing the calculation)?

(C) Give an argument which implies the following claim: 'The speed of a free particle (in the absence of an external potential) constrained to move along an arbitrary, smooth curve is constant.'

Hint: Note that H does not explicitly depend on time, and hence it is Poisson conserved. This allows you to replace H by a conserved energy, $H \to E = mv^2/2$, where v is a conserved speed. Furthermore, you may find useful the following integral,

$$\mathcal{I}_2(\phi) = \int \sqrt{\cosh(2\phi)}d\phi = -\imath \text{EllipticE}(\imath\phi, 2). \tag{6}$$

where $\text{E}(\phi|m) = \text{EllipticE}(\phi, m)$ denotes the *elliptic integral of the second kind*, defined by the integral, $\text{E}(\phi|m) = \int_0^\phi \sqrt{1 - m\sin^2(\tilde{\phi})}d\tilde{\phi}$. This integral has the following small and large argument ϕ expansions,

$$\mathcal{I}_2(\phi)|_{\phi\ll 1} \approx \phi + \frac{\phi^3}{3} - \frac{\phi^5}{30}, \qquad \mathcal{I}_2(\phi)|_{\phi\gg 1} \approx \sqrt{2}\sinh(\phi) - \frac{\Gamma^2(3/4)}{\sqrt{2\pi}}. \tag{7}$$

Problem 3. The Hamiltonian of a Lifshitz model. *(7 points)*
Consider the following scalar field Lagrangian:

$$L[\phi, \dot{\phi}, \nabla\phi] = \int d^d x \mathcal{L}(\phi, \dot{\phi}, \nabla\phi), \qquad \mathcal{L} = \frac{1}{nc_s^n}(\dot{\phi}(x))^n - \frac{1}{2}[\nabla\phi(x)]^2 - V(\phi) \tag{8}$$

where $\mathcal{L}$ is a Lagrangian density, V is a potential, $\phi = \phi(x)$ is a scalar field that depends in general on space and time ($x \equiv (x^0, \vec{x})$, ($x^0 = c_s t$ and c_s is the sound speed), $\nabla = \sum_{i=1}^d \hat{e}^{(i)}\partial_i$ is a gradient operator, $\hat{e}^{(i)}$ are the unit vectors of an orthogonal basis, n is a positive integer, and d is the number of spatial dimensions.

(A) Determine the canonical momentum $\pi_\phi(x)$ of the field ϕ, the Hamiltonian and the Hamiltonian density for the Lifshitz Lagrangian (8).

(B) Calculate the Poisson brackets of the Hamiltonian with ϕ and π_ϕ. Using these brackets show that the Hamilton equations can be written as,

$$[\dot{\phi}(x)]^{n-1} = c_s^n \pi_\phi(x), \qquad \dot{\pi}_\phi(x) = \nabla^2\phi - \frac{d}{d\phi}V(\phi). \tag{9}$$

(C) Does the Lifshitz Lagrangian (8) preserve or break Lorentz symmetry? Justify your answer.

(D) Solve equations (9) for the case when $n = 1$ and for $V = -\rho(\vec{x})\phi(x)$ in $d = 3$, where $\rho(\vec{x})$ is a (static) charge density. Show that, up to a harmonic function, the general solution can be written as,

$$\phi(\vec{x}, t) = \frac{1}{4\pi}\int d^3x' \frac{\rho(\vec{x}\,')}{\|\vec{x} - \vec{x}\,'\|}\,. \tag{10}$$

(E*) *(Bonus question)* Find the solution for $\phi(\vec{x})$ as in part D, but now with $V = (m^2c^2/2\hbar^2)\phi^2(x)$ $-\rho(\vec{x})\phi(x)$, where m is a mass parameter, $\hbar$ is the reduced Planck constant, and c is the speed of light.

Problem 4. Noether's Theorem. *(4 points)*
The Noether theorem of classical field theory states that there are conserved Noether currents associated with a Lagrangian density $\mathcal{L}(\phi_I)$, where $\phi_I(x)$ is a classical field, and they are of the form,

$$J^\mu_n = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I(x))}\Big(\Phi_{I,n} - (\partial_\nu\phi_I)X^\nu_n\Big) - \mathcal{L}X^\mu_n \tag{11}$$

where $n = 1, 2, .., s$ is the index of global coordinate shift parameters ω_n (see Eq. (12) below), and $\Phi_{I,n}$ and X^m_n are defined in terms of the field shifts and the coordinate shifts as

$$\delta\phi_I = \sum_{n=1}^{s}\Phi_{I,n}\omega_n\,, \qquad \delta x^\mu = \sum_{n=1}^{s}X^\mu_n\omega_n\,. \tag{12}$$

Consider now scalar fields, for which $\Phi_{I,n} = 0$, and (global) coordinate shifts of the form,

$$\delta x^\mu = \delta^\mu_\nu\delta\omega^\nu \tag{13}$$

(here a summation over the spacetime index ν is implied, and μ is also a spacetime index) such that $X^\mu_\nu = \delta^\mu_\nu$.

(A) Write down the form of the conserved currents J^μ_ν (in terms of $\mathcal{L}$ and its derivatives) associated with these coordinate shifts. These currents have a name; what is that name?

(B) Consider an integral over all d spatial directions of J^0_ν, where $\nu = (0, i)$ can be a time-like or a spatial index,

$$P_\mu(t) = \int_V d^dx J^0_{\mu}(x)\,. \tag{14}$$

The integral is assumed to be over the whole space, *i.e.* $V \to \infty$. Show that $P_\mu(t)$'s are conserved, *i.e.* $dP_\mu(t)/dt = 0$ ($\mu = (0, i)$). What is the physical significance of P_μ's. Provide a separate interpretation for P_0 and for P_i.

Hint: In order to prove the conservation of P_μ's in part B, make use of the current conservation and of the Gauss' theorem.

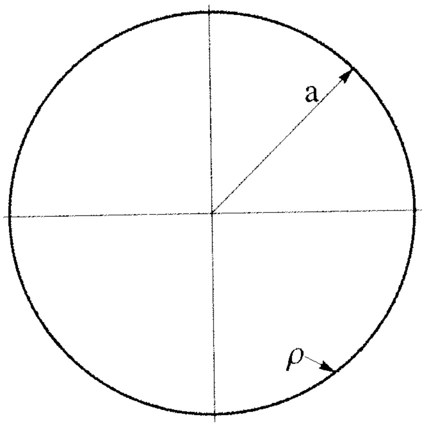
Problem 5. Potential in two spatial dimensions. *(7 points)*

(A) Construct the Green function for two spatial dimensions for an infinite space satisfying,

$$\nabla^2 G(\vec{x};\vec{x}\,') = \nabla'^2 G(\vec{x};\vec{x}\,') = -4\pi\delta^2(\vec{x}-\vec{x}\,')\,. \tag{15}$$

where ∇ and ∇' are the gradient operators with respect to $\vec{x}$ and $\vec{x}'$, respectively.

(B) Make use of the Green function found in part A and of the Poisson equation to solve for the electrostatic potential of a charged ring of radius a (see the figure below). The total charge on the ring is Q. The charge density of the ring is given by $\rho(\vec{x}) = Q\delta^2(\vec{x} - a\hat{\vec{r}}) = [Q/(2\pi a)]\delta(r-a)$, where $\hat{\vec{r}}$ is the unit radial vector.



(C) Make use of the solution in part B to calculate the monopole (q), dipole ($\vec{d}$) and quadrupole moments (q_{ij}) of the problem.

(D*) *(bonus question)*

There are two point static charges q_1 and q_2 ($q_1 q_2 < 0$) and masses m_1 and m_2 at a distance d lying on a plane, such that the two dimensional Green function calculated in part A applies. By solving the relevant equation of motion, calculate the time $t_{\rm coll} = t_{\rm coll}(d)$ as a function of d, q_1, q_2, m_1 and m_2 at which the charges will collide. You will get a power law $t_{\rm coll} \propto d^s$; what is s?

Hint: The Green function in part A is of the form, $G(\vec{x};\vec{x}\,') = -2\ln(\|\vec{x}-\vec{x}\,'\|/r_0))$, where r_0 is some distance scale (changing r_0 corresponds to adding a constant term to the Green function and has thus no physical relevance). For part B, note that the symmetry of the problem tells us that this Green function (with $r_0 = a$ and $\vec{x}\,' = 0$) can be reinterpreted as the potential generated outside the ring with a total unit charge, such that the potential at the ring vanishes. For part D the following integral can be useful,

$$\int_0^1 \frac{dx}{\sqrt{-\ln(x)}} = 2\int_0^\infty e^{-y^2}dy = \sqrt{\pi}\,. \tag{16}$$