Classical Field Theory NS-364B - Final Exam

The exam will take place on Tue Jul 2 at 15-18 hours in BBG 165/169.

The exam is closed books. 90 points in total; counts as 45% of the total grade. Good luck!

July 1, 2013

1 Theoretical questions *(22 points: questions 1-2: 5 pts; 3-4: 6 pts)*

1. Sketch the light cone (on which a massless particle can move) and use it to explain the principle of causality. *Hint: it suffices to draw and explain the allowed trajectories. Emphasize what are the properties that trajectories of physical particles must have.*

2. Lorentz symmetry is a non-compact Lie group. Define a Lie group. What is the non-compactness referring to?

3. Electromagnetism possesses a gauge symmetry. Explain how an electromagnetic vector field A^μ transform under the gauge symmetry. Use this gauge transformation to construct the gauge invariant components of A^μ. Explain also which of these components are dynamical and which are non-dynamical. What is the physical meaning of the non-dynamical component(s)?

4. The Liénard-Wiechert potentials are of the form,

$$\varphi(t,\vec{x}) = \frac{e}{R}, \qquad \vec{A}(t,\vec{x}) = \frac{\vec{v}}{c}\varphi\,. \tag{1.1}$$

   Explain what physical problem the Liénard-Wiechert potentials solve. In which gauge? Explain what is the meaning of $R = \|\vec{R}\|$ and $\vec{v}$ in (1.1). Explain in detail how R depends on t and $\vec{x}$. In what sense are these potentials causal?

2 Is this a classical field? *(24 points; each question 6 pts; bonus 5 pts)*

Consider the following Lagrangian density for two real scalar fields:

$$\begin{aligned}\mathcal{L} &= ic_1\left[\phi_1\dot{\phi}_1 + \phi_2\dot{\phi}_2 - i\phi_2\dot{\phi}_1 + i\phi_1\dot{\phi}_2\right] + \frac{(c_1)^2}{2c_2}\left[(\nabla\phi_1)\cdot(\nabla\phi_1) + (\nabla\phi_2)\cdot(\nabla\phi_2)\right] \\ &\quad - V\left(\phi_1^2+\phi_2^2\right),\end{aligned} \tag{2.1}$$

where V is some potential and the dot represents a time derivative.

1. Calculate the equations of motion for both real fields ϕ_1 and ϕ_2.

2. Introduce a complex scalar field $\psi = \phi_1 + i\phi_2$. Combine the above two equations of motion appropriately to obtain an equation of motion for ψ and set $c_1 = \hbar$ and $c_2 = m$. What do you observe? - *Hint: You should obtain the time-dependent Schrödinger equation for a single non-relativistic particle.*

3. Now, construct an appropriate Lagrangian density for the complex field ψ; if you plug in the *Ansatz* $\psi = \phi_1 + i\phi_2$ into your answer, you should retrieve 2.1). Calculate the corresponding Hamiltonian density.

4. Show that the Lagrangian density for the complex scalar field ψ is invariant under the global $U(1)$ transformation $\psi \to e^{i\alpha}\psi$. Calculate the associated conserved Noether current.

5. *Bonus question:* Comment on why a classical complex scalar field obeys Schrödinger's equation. Do you think quantum mechanics is equivalent to a non-relativistic classical field? Is there any phenomenon in quantum mechanics that you can think of, that you do not expect in Classical Field Theory?

3 Energy momentum tensor *(24pts; each question 4pts; bonus 5pts)*

Consider a collection of charged particles with position $\vec{x}_n(t)$ and charges e_n. The charge density j^0 and the current density $\vec{j}$ are defined as

$$j^0(\vec{x},t) = \sum_n e_n \delta^3(\vec{x} - \vec{x}_n(t)), \qquad \vec{j}(\vec{x},t) = \sum_n e_n \dot{\vec{x}}_n(t)\delta^3(\vec{x} - \vec{x}_n(t)) \tag{3.1}$$

The *energy-momentum tensor* of this system is given by

$$T^{\mu\nu} = \sum_n p_n^\mu(t)\, \dot{x}_n^\nu(t)\, \delta^3(\vec{x} - \vec{x}_n(t))\,. \tag{3.2}$$

1. Show that the energy-momentum tensor is only conserved up to a force density G^μ,

$$\partial_\nu T^{\mu\nu} = G^\mu \tag{3.3}$$

and that G^μ vanishes for free particles (there are no external forces).

2. The electromagnetic force is given by

$$f^\mu \equiv \frac{dp^\mu}{d\tau} = eF^\mu{}_\nu \frac{dx^\nu}{d\tau}\,, \tag{3.4}$$

where for simplicity we took $e_n = e$. Show that the force density G^μ is given by

$$G^\mu = F^\mu{}_\nu j^\nu\,, \tag{3.5}$$

where $j^\mu = (j^0, \vec{j})$.

3. To obtain a conserved energy-momentum tensor, we have to include the contribution of the electromagnetic field itself,

$$T^{\mu\nu}_{em} = F^{\mu\rho}F^{\nu\sigma}\eta_{\rho\sigma} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,. \tag{3.6}$$

Show that $T^{\mu\nu}_{tot} = T^{\mu\nu} + T^{\mu\nu}_{em}$ is conserved. *Hint: Use Maxwell's equations.*

4. Show that the total momentum

$$P^{\mu} = \int d^3x T^{\mu 0}_{tot}(\vec{x}, t) \tag{3.7}$$

is conserved.

Next, we consider the energy momentum tensor of a *perfect fluid.* A comoving (moving along with the fluid) observer will see his/her surroundings as isotropic (this is true only for special configurations of electromagnetic fields, which is what we assume here). In this frame, the particles' energy-momentum tensor (3.2) can be recast into the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{3.8}$$

where ρ and p are the density and pressure of the fluid.

5. Calculate the components ρ and p of the energy-momentum tensor $T^{\mu\nu}$ for an observer at rest. Assume that the comoving observer has a velocity $\vec{v}$ with respect to the observer at rest.

6. Show that the energy-momentum tensor can be written as

$$T^{\mu\nu} = (p + \rho)\frac{U^{\mu}U^{\nu}}{c^2} - p\eta^{\mu\nu}, \tag{3.9}$$

where U^{μ} is the four-velocity of the fluid, which in the fluid rest frame has the form $U^{\mu}_{rest} = (c, 0, 0, 0)$.

7. *Bonus question:* By taking the non-relativistic limit of the conservation law $\partial_{\mu}T^{\mu\nu} = 0$, deduce the following *(Euler's) fluid equations*:

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \tag{3.10}$$

$$\partial_t(\rho\vec{v}) + (\vec{v} \cdot \nabla)(\rho\vec{v}) = -\nabla p. \tag{3.11}$$

Of course, these equations hold only when electromagnetic fields are neglected.

Use that, in the non-relativistic limit, the four-velocity is given by $U^{\mu} = (c, \vec{v})$, where $||\vec{v}|| \ll 1$ and $p \ll \rho$.

4 Advanced Green function *(20 points; each question 10 points)*

1. The positive and negative frequency Wightman functions for a photon vector field (in Lorentz gauge) or a massless scalar field are defined as homogeneous solutions of the wave equation,

$$-\partial_x^2 G^{\pm}(x;x') = 0 = -\partial_{x'}^2 G^{\pm}(x;x')\,, \tag{4.1}$$

whereby (in the vacuum) $G^+(x;x')$ is determined by the contribution obtained by integrating counterclockwise around the positive frequency pole $k^0 = \omega/c = \|\vec{k}\|$, and $G^-(x;x')$ picks up the contribution by integrating clockwise around the negative frequency pole $k^0 = -\omega/c = -\|\vec{k}\|$. Calculate $G^{\pm}$ in position space by performing the suitable 4-momentum integrations and show that

$$G^+(x;x') = \frac{-\imath}{4\pi^2}\frac{1}{\Delta x_+^2}\,, \qquad G^-(x;x') = \frac{-\imath}{4\pi^2}\frac{1}{\Delta x_-^2}\,, \tag{4.2}$$

where $\imath^2 = -1$ and

$$\Delta x_+^2 = (ct - ct' - \imath\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2\,, \qquad \Delta x_-^2 = (ct - ct' + \imath\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2\,. \tag{4.3}$$

Explain the origin of the infinitesimal parameter $\epsilon > 0$ in Eq. (4.3).

2. The Pauli-Jordan, or spectral, two point function can be defined as

$$G_{PJ} = G^- - G^+\,. \tag{4.4}$$

Show that

$$G_{PJ} = -\frac{\mathrm{sign}(t-t')}{2\pi}\delta(\Delta x^2 - \epsilon^2)\,, \tag{4.5}$$

where Δx^2 is defined as $\Delta x^2 = c^2(t-t')^2 - \|\vec{x} - \vec{x}'\|^2$, $\mathrm{sign}(t-t') = \Theta(t-t') - \Theta(t'-t)$ and $\Theta(t-t')$ denotes the Heaviside function. Calculate the advanced Green function in position space,

$$G_a(x;x') = \Theta(t'-t)G_{PJ}(x;x')\,. \tag{4.6}$$

Explain the causality structure of the advanced Green function. Explain also what kind of problems G_a can be used to solve.

Hint: Make use of the Plemelj-Sokhotski theorem,

$$\frac{1}{x \mp \imath\epsilon} = \mathcal{P}\frac{1}{x} \pm \imath\pi\delta(x)\,,$$

where $\mathcal{P}$ denotes a principal value (when integrating) and $\epsilon > 0$ is an infinitezimal parameter.