EXAMINER: DR. UMUT GÜRSOY	UTRECHT UNIVERSITY
DATE: 31/01/2019	FINAL EXAM
TIME: 13:30 - 16:30	

Final exam for Quantum Field Theory

- Write your **name and student number on every sheet**.
- There are 4 problems. Write your answers to the individual problems on different sheets.
- Make sure that your **answers are understandable and readable**. In doubt, explain with a short comment what you are doing.

Some formulas:

- LSZ formula:
$$\langle f|i\rangle = (2\pi)^4\delta^{(4)}(k_{\text{in}} - k_{\text{out}})iT\,,$$
where T is a sum over the relevant Feynman diagrams.
- Pauli matrices:
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
- $\sigma^\mu = (\mathrm{I}, \sigma^i)$, $\quad \bar{\sigma}^\mu = (\mathrm{I}, -\sigma^i)$
- Gamma matrices:
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$
- Chirality operator: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Problem 1: Short questions [20 points]

In this problem we ask you some basic questions concerning the lectures. You should give **short** answers. Do not lose too much time on this problem.

(i) Is chirality of a massive Dirac spinor conserved in time? Explain why or why not. Is helicity conserved? **[4 points]**

(ii) A Dirac spinor has 8 real components which reduce to 4 after use of the equation of motion. What is the physical interpretation of these 4 states? **[3 points]**

(iii) How does a spinor field $\Psi_\alpha(x)$ with α a spin index transform under Lorentz transformations. How does a vector field $A_\mu(x)$ with μ a space-time index transform? **[4 points]**

(iv) Consider the following irreducible representation $(3, 1/2)$ of the Lorentz algebra. What is the dimension of this representation? What are the spin states contained in it? Is this particle a fermion or a boson? **[3 points]**

(v) Consider a scalar interaction term of the form ϕ^n where n is an integer. What is the maximum value of n in 4 dimensions for this interaction to be renormalizable? Motivate your answer. **[3 points]**

(vi) Determine (or write down if you know) the canonical momentum conjugate to the massless vector field A_μ described by the Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. What is your answer for A_0? **[3 points]**

(vii) What are the tadpole diagrams? Explain why they do not contribute to the correlation functions. **Bonus [4 points]**

Problem 2: Scalar Yukawa theory[25 points]

Consider a complex scalar field ϕ and real scalar field φ with Lagrangian density

$$\mathcal{L}_0 = -\partial^\mu \phi^\dagger \partial_\mu \phi - \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - M^2 \phi^\dagger \phi - \frac{1}{2} m^2 \varphi^2 + g \phi^\dagger \phi \varphi \,. \tag{1}$$

(i) What are the Feynman rules for this theory? **[5 points]**

(ii) What are the Feynman diagrams contributing to the four-point function

$$\langle 0 | T \phi^\dagger(x_1) \phi(x_2) \phi^\dagger(x_3) \phi(x_4) | 0 \rangle \tag{2}$$

up to and including order $\mathcal{O}(g^2)$? **[10 points]**

(iii) Calculate the $\phi^\dagger \phi \to \phi^\dagger \phi$ scattering amplitude up to and including order $\mathcal{O}(g^2)$. **[10 points]**

Problem 3: Massive vector field [25 points]

A Lagrangian density for a massive vector field A_μ and real scalar field φ can be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 A_\mu A^\mu + m A^\mu \partial_\mu \varphi \,, \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(i) Is $\mathcal{L}$ invariant under

$$A_\mu \to A_\mu + \partial_\mu \Lambda \,, \tag{4}$$

where Λ is a spacetime dependent function? If not, how should φ transform to get a symmetry of $\mathcal{L}$? **[8 points]**

(ii) Use the symmetry of the previous question to set φ to zero. What is the name of the resulting Lagrangian density? **[5 points]**

(iii) Determine the momentum space propagator for the massive vector field A_μ when $\varphi = 0$. **[12 points]**

Problem 4: Helicity operator [30 points]

Consider the Lagrangian density for the Dirac field

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi\,, \tag{5}$$

for which the general solution of the corresponding equation of motion is given by

$$\Psi(x) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3 2\omega}\left[a_s(\vec{p})u_s(\vec{p})e^{ipx} + b_s^\dagger(\vec{p})v_s(\vec{p})e^{-ipx}\right]\,. \tag{6}$$

The spinors $u_s(\vec{p})$ and $v_s(\vec{p})$ are given by

$$\begin{aligned} u_s(\vec{p}) &= \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi_s \\ \sqrt{-p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} & v_s(\vec{p}) &= \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\eta_s \\ -\sqrt{-p\cdot\bar{\sigma}}\,\eta_s \end{pmatrix}, \\ \xi_1 = \eta_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \xi_2 = \eta_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

After quantization the only non-vanishing anti-commutators are

$$\{a_s(\vec{p}), a_{s'}^\dagger(\vec{k})\} = \{b_s^\dagger(\vec{p}), b_{s'}(\vec{k})\} = (2\pi)^3\,\delta^3(\vec{p}-\vec{k})2\omega\delta_{ss'}\,. \tag{7}$$

The Dirac spin operator is equal to

$$\vec{S} = \frac{1}{2}\int d^3x : \Psi^\dagger\vec{\Sigma}\Psi :, \tag{8}$$

where : : denotes normal ordering which puts annihilation operators to the right and creation operators to the left with a minus sign for each time two operators have to be commuted, i.e. $: a_1(\vec{p})b_2^\dagger(\vec{k}) := -b_2^\dagger(\vec{k})a_1(\vec{p})$. Furthermore we have that

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{9}$$

with $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices.
The helicity operator is then defined by

$$h \equiv \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}. \tag{10}$$

(i) Show that in the frame $\vec{p} = (0, 0, p)$ one can write

$$\begin{aligned} \sqrt{-p\cdot\sigma} &= \frac{1}{2}\sqrt{\omega - p}\left(I + \sigma^3\right) + \frac{1}{2}\sqrt{\omega + p}\left(I - \sigma^3\right), \\ \sqrt{-p\cdot\bar{\sigma}} &= \frac{1}{2}\sqrt{\omega + p}\left(I + \sigma^3\right) + \frac{1}{2}\sqrt{\omega - p}\left(I - \sigma^3\right). \end{aligned} \tag{11}$$

[5 points]

(ii) Working in the frame where $\vec{p} = (0, 0, p)$, show that the helicity operator is given by

$$h = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3\, 2\omega} \sum_{s,r=1}^{2} \left(\sigma^3\right)_{sr} \left[a_s^\dagger(\vec{q})a_r(\vec{q}) - b_s^\dagger(\vec{q})b_r(\vec{q})\right] . \tag{12}$$

[10 points]

(iii) Compute the helicity for the following two-particle states

$$|A\rangle = a_1^\dagger(\vec{p})a_1^\dagger(\vec{k})|0\rangle, \qquad |B\rangle = a_1^\dagger(\vec{p})a_2^\dagger(\vec{k})|0\rangle . \tag{13}$$

[5 points]

(iv) Argue that when $m \neq 0$ the helicity is not Lorentz invariant. **[5 points]**

(v) Use the Dirac equation to show that when $m = 0$, eigenfunctions of chirality (γ^5) with positive energy are also eigenfunctions of helicity ($h = \vec{\Sigma}\cdot\vec{p}/|\vec{p}|$) with the same eigenvalues. **[5 points]**