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In 2005/2006, the course WISB315 was given by Richard D. Gill.

Introduction to Functional Analysis (WISB315)
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Give the reasoning behind your answers and derivations; you can refer to standard results in Saxe's book.

Question 1

Is it true or not true that:

a) $L_2(0,1)$ is a linear subspace of $L_1(0,1)$?

b) $L_2(0,1)$ is a closed linear subspace of $L_1(0,1)$ (with respect to $\|\cdot\|_1$)?

c) Giving $B = C[0,1]$ the L_2 norm, the mapping $T : B \to \mathbb{C}$ defined by $Tf = f(\frac{1}{2})$ is bounded?

d) Giving $B = C[0,1]$ the L_2 norm, the mapping $T : B \to \mathbb{C}$ defined by $Tf = \int_0^1 f(x)\mathrm{d}x$ is bounded?

Question 2

Suppose that $u^{(n)}$, $n = 1, 2, ...$ is a countably infinite orthonormal sequence in a Hilbert space $\mathcal{H}$. Define $\mathcal{U}$ to be the closed linear span of the $u^{(n)}$, i.e., the closure of the set of linear combinations of finitely many $u^{(n)}$.

a) Explain why $\mathcal{U} = \{\sum_n \alpha_n u^{(n)} : \sum_n |\alpha_n|^2 < \infty\}$.

b) For an arbitrary element $v \in \mathcal{H}$ define $A(v) = \sum_n \langle v, u^{(n)}\rangle u^{(n)}$. Explain why $A(v)$ is well defined and is an element of $\mathcal{U}$.

c) We can write $v = z + w$ where $z \in \mathcal{U}$, $w \in \mathcal{U}^\perp$, z and w are unique. We call z the *orthogonal projection of v onto $\mathcal{U}$*. Show that $z = A(v)$ and that A is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. Show that A is Hermitian. Compute its norm and its spectrum. Show that A is not compact.

d) Suppose now that $\mathcal{H} = L_2(-\pi, \pi)$ and take the $u^{(n)}$ to be the sequence of cosine functions, including the constant function, taken from the usual trigonometric basis of $\mathcal{H}$ (i.e., we omit the sines). Define $B : \mathcal{H} \to \mathcal{H}$ by $(B(v))(x) = \frac{1}{2}(v(x) + v(-x))$. Show that $B = A$. Hint: note that any element of $L_2(-\pi, \pi)$ can be written uniquely as a sum of an even and an uneven function: $v(x) = \frac{1}{2}(v(x) + v(-x)) \ + \ \frac{1}{2}(v(x) - v(-x))$.

Question 3

Suppose that f_i, g_i, $i = 1, \ldots, n$ are elements of $C[0,1]$ and define $K(x,y) = \sum f_i(x)g_i(y)$. Suppose the f_i's are linearly independent of one another, and the g_j's are linearly independent of one another. Define Af by $(Af)(x) = \int_0^1 K(x,y)f(y)\mathrm{d}y$.

a) Show that A is a bounded linear operator from $L_2(0,1)$ to $L_2(0,1)$.

b) Describe how you could compute the eigenvalues and eigenvectors of A, and show that its spectrum consists only of eigenvalues. Hint: it may be useful to introduce an orthonormal basis of the linear span of the g_j's and f_i's together. You may express your conclusions in terms of eigenvalues and eigenvectors of a finite dimensional matrix.

Question 4

This exercise concerns the characterization of compact subsets of ℓ_1. (An element u of ℓ_1 is an infinite sequence of numbers u_i such that $\|u\|_1 = \sum_i |u_i| < \infty$).

Show that a subset A of ℓ_1 is compact if and only if it is (i) closed, (ii) bounded, and (iii) *uniformly summable*: for any given $\epsilon > 0$ there exists an i_0 such that for all $u \in A$, $\sum_{i \geq i_0} |u_i| \leq \epsilon$.

You may build up your proof with the following ingredients:

a) Show that a sequence $u^{(n)}$, $n = 1, 2 \ldots.$ of elements of a set A having properties (i)–(iii), has a convergent subsequence (i.e., a subsequence which converges in $\| \cdot \|_1$).

b) Suppose A is closed and bounded but does not satisfy property (iii). That is: there exists an $\epsilon > 0$ such that for each i_0 there exists $u \in A$ with $\sum_{i \geq i_0} |u_i| > \epsilon$. Show there is a sequence $u^{(n)}$, $n = 1, 2, \ldots$ of elements of A without a convergent subsequence.

c) Use the result (a) to show that (i)–(iii) implies A is compact; use result (b) to show that if A does not satisfy (i)–(iii) then it is not compact.