UNIVERSITY OF AMSTERDAM

Faculty of Science

Exam

Measure Theoretic Probability

MasterMath course

Final Exam
Date: February 1st, 2017
Time: 14:00-17:00

Number of pages: 2 (including front page)
Number of questions: 5
Maximum number of points to earn: 20
At each question is indicated how many points it is worth.

BEFORE YOU START

- Please **wait** until you are instructed to open the booklet.
- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and in the coat or bag. Your **coat and bag** must be under your table.
- **Tools allowed**: paper, pen, pencil, eraser.

PRACTICAL MATTERS

- The first 30 minutes and the last 15 minutes you are not allowed to leave the room, not even to visit the toilet.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your enrollment or a valid ID.
- During the examination it is not permitted to visit the toilet, unless the proctor gives permission to do so.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, please fill out the evaluation form at the end of the exam.

Good luck!

Final exam MTP.

Question 1 (5pt) Let (S, Σ, μ) be a measure space, let $h \in \mathcal{L}^1(S, \Sigma, \mu)$ satisfy $h \geq 0$, and let $\nu \colon \Sigma \to \mathbb{R}$ be the measure given by $\nu(A) = \int_A h \, d\mu$.

(a) (3pt) Prove, using the 'standard machinery', that for all $f \in \mathcal{L}^1(S, \Sigma, \nu)$ it holds that $fh \in \mathcal{L}^1(S, \Sigma, \mu)$ and $\int_S f(s) \, d\nu(s) = \int_S f(s)h(s) \, d\mu(s)$.

(b) (2pt) Let $\phi \colon \Sigma \to \mathbb{R}$ be a measure satisfying $\phi \ll \nu$ and let $\frac{d\phi}{d\nu}$ be the Radon-Nikodym derivative of ϕ with respect to ν. Prove that $\phi \ll \mu$ and prove that the Radon-Nikodym derivative $\frac{d\phi}{d\mu}$ of ϕ with respect to μ satisfies $\frac{d\phi}{d\mu} = h \frac{d\phi}{d\nu}$.

Question 2 (2pt) Let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ satisfying $\lim_{n \to \infty} \sigma_n = \sigma$, and let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables such that $Z_n \sim \mathcal{N}(0, \sigma_n)$, $n \in \mathbb{N}$. Prove that there exists a random variable Z_∞ such that $Z_n \xrightarrow{w} Z_\infty$ and $Z_\infty \sim \mathcal{N}(0, \sigma)$.

Question 3 (2pt) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_n)_{n \in \mathbb{N}}$ a filtration on $\mathcal{F}$ and $(M_n)_{n \in \mathbb{N}}$ and $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale. Define $\mathcal{G}_n = \sigma(M_1, \ldots, M_n)$, $n \in \mathbb{N}$. Prove that $(M_n)_{n \in \mathbb{N}}$ is a $(\mathcal{G}_n)_{n \in \mathbb{N}}$ martingale.

Question 4 (7pt) Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{E}(X_1^2) = 1$ and $\mathbb{E}(X_1) = 0$. Let $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ be the filtration given by $\mathcal{F}_0 = \{\emptyset, \mathcal{F}\}$ and, for $n \in \mathbb{N}$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Let $(Y_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$-predictable sequence of integrable random variables. Define a stochastic process $(M_n)_{n \in \mathbb{N}}$ by setting $M_0 = 0$ and, for $n \in \mathbb{N}$, $M_n = \sum_{k=1}^n Y_k X_k$.

(a) (2pt) Prove that $(M_n)_{n \in \mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$-martingale.

(b) (2pt) Prove that if $\mathbb{E}Y_n^2 < \infty$ for all $n \in \mathbb{N}$, then $\mathbb{E}\langle M \rangle_n = \sum_{k=1}^n \mathbb{E}Y_k^2$.

(c) (3pt) **Provide a condition** α on $(Y_n)_{n \in \mathbb{N}}$ such that

i. if a predictable process $(Y_n)_{n \in \mathbb{N}}$ satisfies condition α, then there exists an $M_\infty \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $M_n \to M_\infty$ a.s. and in L^1.

ii. there exists a non-deterministic predicatable process $(Y_n)_{n \in \mathbb{N}}$ satisfying condition α and satisfying $\mathbb{P}(Y_n = 0) < 1$ for all $n \in \mathbb{N}$.

In particular, **give an example** of a non-deterministic predicatable process $(Y_n)_{n \in \mathbb{N}}$ satisfying condition α and satisfying $\mathbb{P}(Y_n = 0) < 1$ for all $n \in \mathbb{N}$.

Question 5 (4pt) Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables satisfying $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) = 1$, and let $\gamma \sim \text{Normal}(0,1)$. Prove[1] that $\frac{1}{n} \sum_{k=1}^n \sqrt{2k-1} X_k \xrightarrow{w} \gamma$ as $n \to \infty$.

[1] *Hint:* the *Lindeberg condition* for a doubly indexed sequence of random variables $(\xi_{n,k})_{n \in \mathbb{N}, k \in \{1,\ldots,n\}}$ reads as follows: *For all* $\varepsilon > 0$ *it holds that*

$$\lim_{n \to \infty} \sum_{k=1}^n \mathbb{E}[\xi_{n,k}^2 \mathbf{1}_{\{|\xi_{n,k}| > \varepsilon\}}] = 0.$$