University of Amsterdam

Faculty of Science

Exam

Measure Theoretic Probability

MasterMath course

Final Exam
Date: January 11th 2017
Time: 14:00-17:00

Number of pages: 2 (including front page)
Number of questions: 4
Maximum number of points to earn: 23
At each question is indicated how many points it is worth.

BEFORE YOU START

- Please **wait** until you are instructed to open the booklet.
- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and in the coat or bag. Your **coat and bag** must be under your table.
- **Tools allowed**: paper, pen, pencil, eraser.

PRACTICAL MATTERS

- The first 30 minutes and the last 15 minutes you are not allowed to leave the room, not even to visit the toilet.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your enrollment or a valid ID.
- During the examination it is not permitted to visit the toilet, unless the proctor gives permission to do so.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, please fill out the evaluation form at the end of the exam.

Good luck!

Final exam MTP.

Question 1 (4pt) For two sets A, B we denote by $A \triangle B$ the *symmetric difference* of A and B, i.e., $A \triangle B := (A \cup B) \setminus (A \cap B)$. Let $\mathcal{B}((0,1)) := \sigma(\{(a,b) : a, b \in [0,1], a < b\})$ be the Borel σ-algebra on $(0,1)$ and let $\lambda \colon \mathcal{B}((0,1)) \to \mathbb{R}$ be the Lebesgue measure on $\mathcal{B}((0,1))$. Prove that for all $A \in \mathcal{B}((0,1))$ it holds that for all $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in [0,1]$ satisfying $a_1 < b_1 < a_2 < \ldots < b_{n-1} < a_n < b_n$ such that $\lambda(A \triangle (\cup_{k=1}^n (a_k, b_k))) < \varepsilon$.

Question 2 (3pt) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra, and let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Formulate (a version of) the Radon-Nikodym theorem and explain how it follows from this theorem that there exists a unique $Y \in L^1(\Omega, \mathcal{G}, \mathbb{P})$ such that for all $A \in \mathcal{G}$ it holds that $\int_A X \, d\mathbb{P} = \int_A Y \, d\mathbb{P}$.

Question 3 (12 pt) Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $\mathbb{P}(X_1 > 0) = 1$, $\mathbb{P}(X_1 = 1) < 1$, $\mathbb{E}(|\ln(X)|) < \infty$ and $\mathbb{E}(X_1) = 1$. Define, for all $n \in \mathbb{N}$, the σ-algebra $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and the random variable $Z_n = \prod_{k=1}^n X_k$.

(a) (1pt) Explain why $\mathbb{E}(\ln(X_1)) \leq 0$.

(b) (1pt) Set $c = \mathbb{E}(\ln(X_1))$. Prove that $(Z_n)^{\frac{1}{n}} \to e^c$ a.s. as $n \to \infty$.

(c) (1pt) Prove that if there exists an $M \in \mathbb{R}$ such that $\mathbb{P}(X_1 \leq M) = 1$, then $(Z_n)^{\frac{1}{n}} \to e^c$ in L^1 as $n \to \infty$.

(d) (2pt) Show that $(Z_n)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale.

(e) (2pt) Show that there exists an $Z_\infty \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $Z_n \to Z_\infty$ a.s. as $n \to \infty$.

(f) (2pt) Prove that there exists an $\varepsilon > 0$ such that $\mathbb{P}(\cap_{n \in \mathbb{N}} \cup_{m \geq n} \{|X_m - 1| > \varepsilon\}) = 1$.

(g) (2pt) Prove that on $\{Z_\infty > 0\}$ it holds that $\lim_{n \to \infty} X_n = 1$, and use this and part (f) to conclude that $\mathbb{P}(Z_\infty = 0) = 1$.

(h) (1pt) Is $(Z_n)_{n \in \mathbb{N}}$ uniformly integrable? (Explain your answer.)

Question 4 (4pt) Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables with characteristic function ϕ. For $n \in \mathbb{N}$ let B_n be a Bernoulli-$(n, \frac{1}{n})$ distributed[1] random variable independent of $(X_k)_{k \in \mathbb{N}}$ and let

$$S_n = \begin{cases} 0, & B_n = 0; \\ \sum_{k=1}^{B_n} X_k, & \text{otherwise.} \end{cases}$$

(a) (2pt) Let ψ_n denote the characteristic function of S_n, $n \in \mathbb{N}$. Prove that for all $s \in \mathbb{R}$ it holds that $\psi_n(s) = \left(1 + \frac{\phi(s) - 1}{n}\right)^n$.

(b) (2pt) Explain[2] why there exists a random variable S_∞ such that $S_n \xrightarrow{w} S_\infty$ and provide the characteristic function of S_∞.

[1]This means that for all $k \in \{0, 1, \ldots, n\}$ it holds that $\mathbb{P}(B_n = k) = \binom{n}{k} n^{-n} (n-1)^{n-k}$.

[2]*Hint:* for all $z \in \mathbb{C}$ it holds that $\lim_{n \to \infty} (1 + \frac{z}{n})^n = e^z$.

$\phi(u) = \int_{\mathbb{R}} e^{ixu} \, d\mu(x)$

Exercise 1:

Let P_ϵ be the property defined in the exercise: $A \subset (0,1)$ satisfies P_ϵ if and only if there exists a $n \in \mathbb{N}$ and $0 \leq a_1 < b_1 < \ldots < a_n < b_n \leq 1$ such that $\lambda(A\Delta(\cup(a_i,b_i))) < \epsilon$. Using this, define the collection:

$$\mathcal{A} := \{A \subset (0,1) \,|\, \forall\epsilon : A \text{ satisfies } P_\epsilon\}$$

Clearly, the exercise wants us to show that $\mathcal{B}(0,1) \subset \mathcal{A}$. We will do this by showing that $\mathcal{A}$ is a $\sigma-$algebra which contains the generators (a,b) of $\mathcal{B}(0,1)$. The usual argument shows then that the whole of $\mathcal{B}(0,1)$ is contained in $\mathcal{A}$

- Let $A = (a,b)$ be an element of the generating set of $\mathcal{B}(-1,1)$. Then, picking $n=1$ and $a_1 = a$, $b_1 = b$ shows that $\lambda(A\Delta(a_1,b_1)) = \lambda(\emptyset) = 0 < \epsilon$ for all ϵ. I.e., A satisfies P_ϵ for all ϵ and $A \in \mathcal{A}$.

- Let $A = (0,1)$. Note that this is a special case of the previous point and hence $(0,1) \in \mathcal{A}$

- Let $A \in \mathcal{A}$. First note that for any set $B \subset (0,1)$ we have that

  $$A^c\Delta B^c = (A^c \cup B^c) \setminus (A^c \cap B^c) = (A \cap B)^c \setminus (A \cup B)^c = (A \cup B) \setminus (A \cap B) = A\Delta B$$

  Let $\epsilon > 0$ be given. As $A \in \mathcal{A}$ it satisfies P_ϵ. Therefore, there exists $B = (a_1,b_1) \cup \ldots \cup (a_n,b_n)$ such that $\lambda(A\Delta B) < \epsilon$. Note that the interior of B^c is given by $(0,a_1) \cup (b_1,a_2) \cup \ldots \cup (b_n,1) =: C$ and is of the form used in the definition of P_ϵ. As C and B^c only differ by a finite amount of points, which have measure zero, we get that $\lambda(A^c\Delta C) = \lambda(A^c\Delta B^c) = \lambda(A\Delta B) < \epsilon$. That is A^c satisfies P_ϵ for all ϵ and hence $A^c \in \mathcal{A}$.

- Let $A_1, A_2 \in \mathcal{A}$, we will show that $A := A_1 \cup A_2 \in \mathcal{A}$. Note that this is not enough to show that $\mathcal{A}$ is a $\sigma-$algebra, as it only proves finite unions, but we will use it to show the same statement for a countable union. Let $\epsilon > 0$ be given. As $A_1, A_2 \in \mathcal{A}$, there exist $n_i \in \mathbb{N}$ and $B_i = (a_1^i, b_1^i) \cup \ldots \cup (a_{n_i}^i, b_{n_i}^i)$ such that $\lambda(A_i\Delta B_i) < \frac{\epsilon}{2}$ for $i = 1,2$. Note that $B := B_1 \cup B_2$ is also a finite union of open intervals. Now note:

  $$\begin{aligned} A\Delta B &= (A_1 \cup A_2 \cup B_1 \cup B_2) \setminus ((A_1 \cup A_2) \cap (B_1 \cup B_2)) \\ &\subset (A_1 \cup A_2 \cup B_1 \cup B_2) \setminus ((A_1 \cap B_1) \cup (A_2 \cap B_2)) \\ &\subset A_1\Delta B_1 \cup A_2\Delta B_2 \end{aligned}$$

  Hence, we find that $\lambda(A\Delta B) \leq \lambda(A_1\Delta B_1) + \lambda(A_2\Delta B_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. We conclude that A satisfies P_ϵ for all ϵ and hence $A \in \mathcal{A}$. By using an induction argument, or just redoing this proof with some more indices, we conclude that any finite union of sets of $\mathcal{A}$ belong to $\mathcal{A}$.

- Let $A_1, A_2, \ldots \in \mathcal{A}$ and let $A = \cup_i A_i$. By replacing A_2 with $(A_2^c \cup A_1)^c$, which lies is $\mathcal{A}$ by the previous two points, we can assume that all the A_i are mutually disjoint. Now, let $\epsilon > 0$ be given. As $A \subset (0,1)$, we find that $\lambda(A) \leq \lambda(0,1) = 1$ and hence there exists a $N \in \mathbb{N}$ such that $\lambda(A^\infty := \cup_{i=N}^\infty A_i) < \frac{\epsilon}{2}$. Also, by the previous point, we realize that $A^N := \cup_{i=1}^{N-1} A_i \in \mathcal{A}$ and hence there exists a $n \in \mathbb{N}$ and $B = (a_1,b_1) \cup \ldots \cup (a_n,b_n)$ such that $\lambda(A^N\Delta B) < \frac{\epsilon}{2}$. Now, we find that:

  $$\begin{aligned} A\Delta B = (A \cup B) \setminus (A \cap B) &\subset (A^N \cup A^\infty \cup B) \setminus (A^N \cap B) \\ &\subset A^\infty \cup ((A^N \cup B) \setminus (A^N \cap B)) = A^\infty \cup A^N\Delta B \end{aligned}$$

Hence, we get that $\lambda(A \triangle B) \leq \lambda(A^{\infty}) + \lambda(A^N \triangle B) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. We conclude that A satisfies P_ϵ for any ϵ and hence $A \in \mathcal{A}$.

1 Problem 2

RN theorem: Let μ be a positive σ-finite measure and let ν be a complex measure. Then, there exist unique ν_a, ν_s such that $\nu = \nu_a + \nu_s$ and a function $h \in L^1(S, \sigma, \mu)$ such that

$$\nu_a(E) = \mu(1_E h)$$

for all $E \in \Sigma$ and $\nu_s \perp \mu$. Moreover, h is μ-a.s. unique.
For the second part define $\nu^{\pm}$ on $\mathcal{G}$ by setting for all $A \in \mathcal{G}$

$$\nu^{\pm}(A) = \int_A X^{\pm} d\mathbb{P}.$$

These are two finite positive measures that are absolutely continuous wrt $\mathbb{P}$. By RN Theorem there exist two $\mathcal{G}$-measurable functions $h^{\pm} : \Omega \to [0, \infty)$ such that for every $A \in \mathcal{G}$

$$\nu^{\pm}(A) = \int_A h^{\pm} d\mathbb{P}.$$

Define $Y := h^+ - h^-$. The uniqueness is trivial.

2 Problem 3

A. Since $\ln x$ is a concave function we can apply Jensen inequality to see that $\mathbb{E} \ln X_1 \leq \ln \mathbb{E} X_1 = 0$.
B. Since $\ln X_n$ are iid by the Strong Law of Large Numbers we have an a.s. convergence:

$$\frac{1}{n} \sum_{k=1}^{n} \ln X_k \to \mathbb{E} \ln X_1 = c.$$

Then, by Proposition 7.6 we also have that $Z_n^{1/n} = e^{\frac{1}{n} \sum_{k=1}^{n} \ln X_k} \to e^c$ almost surely.
C. Given the assumption, we have $\mathbb{P}(0 \leq e^{\frac{1}{n} \sum_{k=1}^{n} \ln X_k} \leq e^{\ln M}) = 1$. Then we can apply Dominated Convergence Theorem to interchange the integral and the limit:

$$\lim_{n \to \infty} \int |Z_n^{1/n} - e^c| d\mathbb{P} = \int \lim_{n \to \infty} |Z_n^{1/n} - e^c| d\mathbb{P}.$$

In the view of B we obtain the required L^1 convergence.
D. Clearly, Z_n is $\mathcal{F}_n$ adapted. Also,

$$\mathbb{E}|Z_n| = \prod_{k=1}^{n} \mathbb{E} e^{\ln X_k} = (\mathbb{E} X_k)^n = 1.$$

Hence, $Z_n \in L^1$.
Finally, since X_{n+1} is independent of $\{X_k\}_{k=1}^{n}$ and for $i = 1 \ldots n$ X_i is $\mathcal{F}_n$-measurable, we have

$$\mathbb{E}(Z_{n+1} \,|\, \mathcal{F}_n) = \mathbb{E}\left(e^{\ln X_{n+1}} \prod_{k=1}^{n} e^{\ln X_k} \,|\, \mathcal{F}_n\right) = Z_n \mathbb{E}\left(e^{\ln X_{n+1}} \,|\, \mathcal{F}_n\right) = Z_n \mathbb{E} X_{n+1} = Z_n.$$

Thus, Z_n is a martingale.
E. Since $\sup_n \mathbb{E}|Z_n| = 1 < \infty$ we can apply Theorem 10.5 to see that there exists $Z_\infty \in L^1$ such that $Z_n \to Z_\infty$ almost surely.
F. Since X_i are iid we also have the independence of A_n. Also, $\mathbb{P}(A_n) = \mathbb{P}(A_1) = c > 0$ for every $n \in \mathbb{N}$. Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} c = \infty.$$

Now we can apply the BK lemma to obtain the required result: $\mathbb{P}(\bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} A_m) = 1$.
G. Since $Z_n \to Z_\infty$ on the set $\{Z_\infty > 0\}$ we have $\lim_n \sum_{k=1}^{n} \ln X_k = \ln Z_\infty$. Hence, $\lim_{n \to \infty} \ln X_n = 0$, which implies $\lim_{n \to \infty} X_n = 1$.
By part F we have that $\mathbb{P}(X_n \to 1) = 0$. Using the first part of G and the fact that $Z_\infty \geq 0$ we conclude that $\mathbb{P}(Z_\infty = 0) = 1$.
H. No. Assume that Z_n is UI. Then by Theorem 10.8 we have $\mathbb{E}(Z_\infty \,|\, \mathcal{F}_n) = Z_n$ almost surely. By part G it would mean that Z_n=0 a.s. for every n, which is clearly not the case, since $\mathbb{P}(X_1 > 0) = 1$.

3 Problem 4

A. Since B_n is independent of X_n, we have that

$$\begin{aligned}\psi_n(s) =& \mathbb{E}e^{isS_n} = \sum_{m=0}^{n} \binom{n}{m} n^{-n}(n-1)^{n-m}\mathbb{E}e^{is\sum_{k=1}^{m} X_k} = \\ =& \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\phi(s)}{n}\right)^m \left(1-\frac{1}{n}\right)^{n-m} = \left(\frac{\phi(s)}{n}+1-\frac{1}{n}\right)^n\end{aligned}$$

B. By the reminder

$$\left(\frac{\phi(s)-1}{n}+1\right)^n \to e^{\phi(s)-1}.$$

We have seen in class that the right hand side is the characteristic function of a random variable $Y := \sum_{k=1}^{N} X_k$, where N is a Poisson(1) rv independent of a sequence X_n. Then by Corollary 13.14 we have a weak convergence of S_n to Y.