Mastermath	**Commutative algebra exam**	**14:00-17:00 19/2/2019**

- Write your name, university, and student number on every sheet you hand in.
- You may not use any books or notes during the exam.
- Unless stated otherwise, you need to give full proofs in all your answers. You are allowed to use results that are treated in the book and lectures.
- If you cannot do a part of a question, you may still use its conclusion later on.

(1) Let A be a ring.

(a) Let $0 \to M'' \xrightarrow{f} M \xrightarrow{g} M' \to 0$ be an exact sequence of A-modules. Show that if both M'' and M' are finitely generated then M is finitely generated.

(b) Let M and N be flat A-modules. Show that $M \otimes_A N$ is flat as an A-module.

(2) Let A be a Noetherian ring. Prove that the following are equivalent:

(a) A is Artinian;

(b) $\mathrm{Spec}(A)$ is discrete and finite;

(c) $\mathrm{Spec}(A)$ is discrete.

(3) Let A be a ring.

(a) Give the definition of the *Jacobson Radical* of A.

(b) State Nakayama's Lemma for finitely generated modules over A.

Now let A be a local ring with maximal ideal $\mathfrak{m}$, and $f\colon M \to N$ a map of finitely-generated A-modules. We write $f_{\mathfrak{m}}$ for the induced map $M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$.

(c) Assume that M and N are finitely generated as A-modules, and that $f_{\mathfrak{m}}$ is surjective. Show that f is surjective.

(d) Show that part (c) fails without the 'finitely generated' hypothesis. More precisely, give an example of a local ring A and a map $f\colon M \to N$ of A-modules which is *not* surjective, but such that the induced map $M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$ *is* surjective.

(4) Let A be the localisation of the ring $R = \frac{\mathbb{Q}[x,y,z]}{xy-z^3}$ at the maximal ideal (x, y, z).

(a) Show that A is Noetherian and local.

(b) What is the dimension of A?

(c) Is A Artin?

(d) Is A regular?

(5) Let A be a ring.

(a) Give the definition of the *nilradical* of A.

(b) Prove Proposition (1.8), that the nilradical of A is equal to the intersection of all the prime ideals in A. You may use results from the book before (1.8). If you use Zorn's lemma, please check the criteria for it carefully.