- There are 4 hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each problem.
- Clearly write DRAFT on any draft page you hand in.

MOAWOA :: SOLUTIONS

May 13, 2016

Problem 1. An invertible 2×2-matrix M with real entries is called a *MOAWOA-matrix* if its inverse M^{-1} can be obtained by permuting the entries of M. Show that if M is a MOAWOA-matrix, then so is M^2.

Proposed by Merlijn Staps (Universiteit Utrecht).

Solution. Write $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and let $D = \det M$. Then we have $M^{-1} = \frac{1}{D}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. Now, suppose M is a MOAWOA-matrix. Then M^{-1} can be obtained by permuting the entries of M. Therefore, we must have $|a|+|b|+|c|+|d| = |\frac{d}{D}| + |\frac{-b}{D}| + |\frac{-c}{D}| + |\frac{a}{D}| = \frac{|a|+|b|+|c|+|d|}{|D|}$. Since M is invertible, we have $|a|+|b|+|c|+|d| > 0$. It follows that $|D| = 1$, hence $D = \pm 1$. First suppose $D = 1$. Then we have $[a, b, c, d] = [d, -b, -c, a]$, where we use square brackets to denote multisets. We find $[b, c] = [-b, -c]$, which implies that $b = -c$. We therefore have $M = \left(\begin{smallmatrix} a & b \\ -b & d \end{smallmatrix}\right)$ with $ad + b^2 = 1$. Conversely, any matrix of this form is a MOAWOA-matrix. In particular, since $\det(M^2) = 1$ and $M^2 = \left(\begin{smallmatrix} a^2-b^2 & ab+bd \\ -ba-bd & -b^2+d^2 \end{smallmatrix}\right)$ the matrix M^2 is a MOAWOA-matrix. Now suppose $D = -1$. Then we have $[a, b, c, d] = [-d, b, c, -a]$. It now follows that $a = -d$. Therefore, $M = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$. We find $M^2 = \left(\begin{smallmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ since $a^2 + bc = -\det M = 1$. Clearly, M^2 is a MOAWOA-matrix. $\square$

Problem 2. Suppose I and J are (real) open intervals of finite positive length, each interval not containing the other. Show that there exists a $\lambda \neq 0$ such that $x \mapsto e^{\lambda x}$ maps I and J to intervals of equal length if and only if I and J have different lengths.

Proposed by Leslie Molag (Katholieke Universiteit Leuven).

Solution. Denote the endpoints of I and J by $a < b$ and $c < d$ respectively. Without loss of generality $d > b$ and $c > a$ (since each interval does not contain the other). Let us define the function

$$f(\lambda) = \begin{cases} \frac{d-c}{b-a} & \text{if } \lambda = 0, \\ \frac{e^{d\lambda}-e^{c\lambda}}{e^{b\lambda}-e^{a\lambda}} & \text{otherwise.} \end{cases}$$

Note that I and J being mapped to intervals of equal length by $x \mapsto e^{\lambda x}$ is equivalent to f attaining the value 1 in some $\lambda \neq 0$. The function f is continuous by construction. We notice that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. When we assume that I and J have distinct lengths we also know that $f(0) \neq 1$, thus by the intermediate value theorem there exists an $\lambda \neq 0$ such that $f(\lambda) = 1$. When I and J have the same length we have $f(\lambda) = e^{(c-a)\lambda}$, which does not equal 1 for any $\lambda \neq 0$. $\square$

Problem 3. Consider n people that stand in a circle. Initially, each of them holds a red and a blue ball. In a turn, each person gives one of his balls to his right neighbor and his other ball to his left neighbor. Does there exist a sequence of turns (starting from the initial situation) such that every possible color distribution of the balls occurs exactly once

(a) if $n = 2016$?

(b) if $n = 2015$?

Based on a proposal by Wouter Zomervrucht (Freie Universität Berlin).

Solution. Number the people $1, \dots, n$ in circular order. We denote by $[a_1 a_2 \cdots a_n]$ the color distribution where person i has precisely a_i blue balls (and $2 - a_i$ red balls). For a word w consisting of the letters 0, 1 and 2 we write w^k for the word that is obtained by concatenating k copies of w.

For (a), the answer is "no". Consider the color distributions $[(20)^{1008}]$ and $[(02)^{1008}]$. Note that these distributions can only be reached from each other. This means that they cannot occur in a sequence of turns that starts with the initial distribution.

For (b), the answer is also "no". Consider the color distribution $C = [(2200)^{503}210]$. The only possible neighbors of this color distribution in a sequence of turns are $[(1)^{2012}012]$ and the initial distribution. Suppose a sequence of turns exists in which each distribution occurs exactly once. Then C can occur only as the second or the last distribution in this sequence. However, the same holds for all cyclic permutations of C, of which there are 2015. This is a contradiction. □

Problem 4. We consider sequences $a_0, a_1, a_2, \dots$ of real numbers that satisfy

$$a_n = 4a_{n-1}(1 - a_{n-1})$$

for all positive integers n. How many such sequences satisfy $a_{2016} = a_0$?

Proposed by Merlijn Staps (Universiteit Utrecht).

Solution. There are 2^{2016} such sequences. If $a_0 < 0$ we have $a_1 = 4a_0(1 - a_0) < a_0$ because $4(1 - a_0) > 1$. It then follows that $a_{2016} < a_{2015} < \cdots < a_1 < a_0$, so we cannot have $a_{2016} = a_0$. If $a_0 > 1$ we have $a_1 = 4a_0(1 - a_0) < 0$ and it follows that $a_{2016} < 0 < a_0$. Hence if $a_{2016} = a_0$ we must have $a_0 \in [0, 1]$. This means that we can write $a_0 = \sin^2(\alpha)$ for some $\alpha \in [0, \frac{\pi}{2}]$. If $a_{n-1} = \sin^2(\beta)$ we have $a_n = 4a_{n-1}(1 - a_{n-1}) = 4\sin^2(\beta)(1 - \sin^2(\beta)) = 4\sin^2(\beta)\cos^2(\beta) = (2\sin(\beta)\cos(\beta))^2 = \sin^2(2\beta)$. By induction, it follows that $a_n = \sin^2(2^n\alpha)$ for all $n \geq 0$. In particular, we have $a_{2016} = \sin^2(2^{2016}\alpha)$. From $a_0 = a_{2016}$ it now follows that $2^{2016}\alpha = \pm\alpha + k\pi$ where k is an integer. This means that $\alpha = \pi \cdot \frac{k}{2^{2016} \pm 1}$. For $\alpha = \frac{k\pi}{2^{2016}-1}$ we must have $0 \leq k \leq \frac{2^{2016}-1}{2} = 2^{2015} - \frac{1}{2}$, which is satisfied for 2^{2015} values of k. For $\alpha = \frac{k\pi}{2^{2016}+1}$ we must have $0 \leq k \leq \frac{2^{2016}+1}{2} = 2^{2015} + \frac{1}{2}$, which is satisfied for $2^{2015} + 1$ values of k. Because $2^{2016} + 1$ and $2^{2016} - 1$ are coprime only the value $\alpha = 0$ is counted twice, so in total there are $2^{2015} + (2^{2015} + 1) - 1 = 2^{2016}$ possible values for α. This means that there are also 2^{2016} possible sequences. □

Problem 5. We are given N weights, with masses 1 kg, 2 kg, ..., N kg. We want to select at least two of these weights, such that their total mass equals the average mass of the other weights. Show that this is possible if and only if $N+1$ is a square.

Proposed by Arne Smeets (Katholieke Universiteit Leuven).

Solution. Suppose we can select weights such that the condition holds. Let $k \geq 2$ be the number of selected weights and let S be the sum of their masses. Then we must have $N \geq S \geq 1+2+\ldots+k = \frac{k(k+1)}{2}$. Furthermore, we have $S = \frac{\frac{N(N+1)}{2}-S}{N-k}$, which rewrites to $2S(N-k+1) = N(N+1)$. It follows that $N-k+1$ divides $N(N+1)$, hence it also divides

$$N(N+1) - (N+k)(N-k+1) = k(k-1).$$

We have $k(k-1) \leq 2N-2k < 2(N-k+1)$, so we must have $N-k+1 = k(k-1)$ and $N+1 = k^2$.
Conversely, if $N+1 = k^2$ then we can select the weights with masses 1 kg, ..., k kg. □

Problem 6. Let k be a positive integer. We consider all possible football matches in which $2k$ goals are scored in total. Show that the number of such matches in which the end result is a draw equals the number of such matches in which the home team is never behind. (By a "football match" we mean the set of all intermediate scores that occur during the match.)

Proposed by Raymond van Bommel and Julian Lyczak (Universiteit Leiden).

Solution. Define $B(x,y) = \binom{x+y}{x} - \binom{x+y}{x+1}$ for all integers $x, y \geq 0$ (if $m > n$, we let $\binom{n}{m} = 0$). Now we see for all $x \geq 0$ that $B(x, x+1) = \binom{2x+1}{x} - \binom{2x+1}{x+1} = 0$ and $B(x,0) = \binom{x}{x} - \binom{x}{x+1} = 1$. For all $1 \leq y \leq x$ we find that

$$\begin{aligned} B(x,y) &= \binom{x+y}{x} - \binom{x+y}{x+1} \\ &= \binom{x+y-1}{x-1} + \binom{x+y-1}{x} - \binom{x+y-1}{x} - \binom{x+y-1}{x+1} \\ &= B(x-1,y) + B(x,y-1). \end{aligned}$$

From this recursion we now see that for $0 \leq y \leq x$, the number $B(x,y)$ also indicates the number of matches that ends in (x,y) and for which the home team is never behind. The number of these matches in which $2k$ goals are scored is therefore

$$\sum_{i=0}^{k} B(k+i, k-i) = \sum_{i=0}^{k} \left[\binom{2k}{k+i} - \binom{2k}{k+i+1} \right] = \binom{2k}{k} - \binom{2k}{2k+1} = \binom{2k}{k},$$

which is equal to the number of matches with $2k$ goals that end in a draw. □

Alternative solution 1. We represent possible matches by lattice paths. A goal for the home team is represented by a step in northeast direction, whereas a goal for the away team is represented by a step in southeast direction. The *height* of a certain point in the path is defined as the difference between the number of steps in northeast direction and the number of steps in southeast direction until that point. The height of the last point of the path is called the *path height*. The *depth* of a path is defined as the lowest height that occurs.
We will construct a bijection between the collection G_n of paths with length $2n$ and path height 0 and the collection B_n of paths with length $2n$ with depth 0. Consider a path in G_n with

depth $-d \leq 0$. Let P be the last point for which height $-d$ occurs. We now reverse the order of the path before P and leave the remainder of the path unchanged. This yields a new path, of which we claim that it is an element of B_n. Indeed, the reversed part has nonnegative height everywhere and has a path height of d. The second part has height at least d everywhere. We observe that the obtained path has $2n$ steps and height 0, so it is in B_n. Furthermore, we note that the path has final height $2d$.
Conversely, a path in B_n has height $2h$ for some $h \geq 0$. The point P can now be found as the last point for which the path has height h. By reversing the order of the path prior to this point, we obtain the inverse of the construction outlined above. Therefore, the construction yields a bijection between B_n and G_n.
It follows that the number of matches $|G_k|$ that end in a draw is equal to the number of matches $|B_k|$ in which the home team is never behind. $\square$

Alternative solution 2. The number of matches for which the end result is a draw equals $\binom{2k}{k}$. The number of matches for which the home team is never behind can be counted by counting paths in $\mathbb{Z}^2$ that consist of steps north and east. Let f_n be the number of such paths that start in the origin, never rise above the diagonal $y = x$ and contain $2n$ steps in total. Then it suffices to show that $f_n = \binom{2n}{n}$. By removing the last step of such a path and replacing it by an initial step east we obtain a path with $2n$ steps that stays strictly below the diagonal. Therefore, there are $\frac{f_n}{2}$ paths of $2n$ steps that stay strictly below the diagonal. Similarly, there are $\frac{f_n}{2}$ paths of $2n$ steps that stay strictly above the diagonal. We conclude that there are $\frac{f_n}{2} + \frac{f_n}{2} = f_n$ paths of $2n$ steps that only touch the diagonal at the origin.
For $|x| < \frac{1}{4}$ we have $\frac{1}{\sqrt{1-4x}} = \sum_{k\geq 0} \binom{2k}{k} x^k$. It follows that $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$ is the n-th coefficient of $\left(\frac{1}{\sqrt{1-4x}}\right)^2$, which equals 4^n. We obtain the identity $4^n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$ for all $n \geq 0$. It now suffices to prove that we also have $4^n = \sum_{k=0}^{n} \binom{2k}{k} f_{n-k}$ since this recursion uniquely determines the sequence (f_n). The left hand side counts lattice paths consisting of $2n$ steps. The summand for k on the right counts the number of such paths that pass through (k, k) but no higher point on the diagonal. The proof is complete. $\square$

Remark. By combining the two alternative solutions, one obtains a combinatorial proof of the identity $4^n = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}$.